On a conjecture of Naito-Sagaki: Littelmann paths and Littlewood-Richardson Sundaram tableaux

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Abstract. In recent work with Schumann we have proven a conjecture of Naito-Sagaki giving a branching rule for the decomposition of the restriction of an irreducible representation of the special linear Lie algebra to the symplectic Lie algebra, therein embedded as the fixed-point set of the involution obtained by the folding of the corresponding Dynkin diagram. This conjecture had been open for over ten years, and provides a new approach to branching rules for non-Levi subalgebras in terms of Littelmann paths. In this extended abstract we motivate the conjecture, prove it for several cases, where we also relate it to the combinatorics of polytopes and Littlewood-Richardson cones, and highlight some difficulties of the proof in general.

Résumé. Dans un travail récent avec Schumann, nous avons demontré une conjecture de Naito-Sagaki donnant une règle de branchement pour la restriction d'une représentation irréductible de l'algèbre de Lie spéciale linéaire à l'algèbre de Lie symplectique, qui se plonge comme les points fixes de l'automorphisme de pliage du diagramme de Dynkin. Cette conjecture a été ouverte depuis plus de dix ans, et donne une nouvelle perspective sur les règles de branchement pour les sous-algèbres de Lie qui ne sont pas de Levi, en termes des chemins de Littelmann. Dans ce résume étendu, nous motivons la conjecture et la démontrons dans certains cas, pour lesquels nous faisons aussi un lien avec la combinatoire des polytopes et cônes de Littlewood-Richardson ; et nous décrivons les difficultés de la démonstration du cas général.

Keywords: Littelmann paths, tableaux combinatorics, branching rules

1 Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field of complex numbers and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an automorphism of finite order. In [5], Kac has shown that σ is conjugate to the automorphism induced by a Dynkin diagram automorphism, and that the set of fixed points $\mathfrak{g}^{\sigma}$ is again a semi-simple Lie algebra, of one of six possible types. The basic problem of decomposing the restriction of an irreducible representation of $\mathfrak{g}$ to $\mathfrak{g}^{\sigma}$ (known as a "branching rule") is, in general, open.

If, however, the automorphism σ is semi-simple and has infinite order, then $\mathfrak{g}^\sigma$ is a Levi subalgebra [15]. In this case, there exist natural and beautiful branching rules in terms of Littelmann paths [7]. Littelmann paths are generalisations of Young tableaux to all Kac-Moody Lie algebras, and were used [7] to give a remarkable generalisation of the Littlewood-Richardson rule for the decomposition of the tensor product of two irreducible finite-dimensional representations.

In [9] a new approach to the branching problem was suggested by Naito-Sagaki in terms of Littelmann paths for $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$ and $\mathfrak{g}^\sigma = \mathfrak{sp}(2n, \mathbb{C})$ in the form of a conjecture. In this paper we reformulate the conjecture in terms of symplectic Littlewood-Richardson tableaux, which were previously studied by Sundaram in [12]. We call them Littlewood-Richardson Sundaram tableaux. We give a proof of the conjecture in some cases using this reformulation, which was used in [11] to give a general proof. Our methods imply a bijection between lattice points of polytopes, in the spirit of Berenstein-Zelevinsky and Knutson-Tao [1] [6]. We believe that this new approach to branching will play an important role in the future.

2   Notation for the Lie algebras

Let $\mathfrak{h} \subset \mathfrak{sl}(2n, \mathbb{C})$ be the Cartan sub-algebra of diagonal matrices. Let ε_i be the linear map $\mathfrak{h} \to \mathbb{C}$ defined by $\operatorname{diag}(a_1, \cdots, a_{2n}) \mapsto a_i$. We write $\mathfrak{sl}(2n, \mathbb{C}) = \langle x_i, y_i, h_i \rangle_{i \in \{1, \cdots, 2n-1\}}$ where $h_i = \mathrm{E}_{ii} - \mathrm{E}_{i+1,i+1}$ and where x_i and y_i are the Chevalley generators corresponding to the simple root $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The automorphism σ is given by

$$\sigma(x_i) = x_{2n-i},$$
$$\sigma(y_i) = y_{2n-i}, \text{ and}$$
$$\sigma(h_i) = h_{2n-i}.$$

The fixed point set $\mathfrak{g}^\sigma$ is generated as a Lie algebra by $\langle \hat{x}_i, \hat{y}_i, \hat{h}_i \rangle_{i \in \{1, \cdots, n\}}$ (see Proposition 7.9 in [5]), where

$$\hat{x}_i = \begin{cases} x_i + x_{2n-i} & \text{if } i \in [1, n) \\ x_n & \text{if } i = n \end{cases}$$

$$\hat{y}_i = \begin{cases} y_i + y_{2n-i} & \text{if } i \in [1, n) \\ y_n & \text{if } i = n \end{cases}$$

$$\hat{h}_i = \begin{cases} h_i + h_{\alpha_{2n-i}} & \text{if } i \in [1, n) \\ h_n & \text{if } i = n. \end{cases}$$

This Lie algebra is isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$ (see Proposition 7.9 in [5]) and $\mathfrak{h}^\sigma = \bigoplus_{i=1}^{n} \hat{h}_i = \mathfrak{h} \cap \mathfrak{g}^\sigma \subset \mathfrak{h}$ is a Cartan subalgebra. Let $\mathrm{P}_{C_n} \subset (\mathfrak{h}^\sigma)^*$ be the set of integral weights of $\mathfrak{g}^\sigma$ with respect to $\mathfrak{h}^\sigma$, and let $\mathrm{P}_{A_{2n-1}} \subset \mathfrak{h}^*$ be the set of integral weights of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

To avoid confusion we will denote elements of $P_{A_{2n-1}}$ by λ and elements of P_{C_n} by $\hat{\lambda}$; in particular, the fundamental weights in $P_{A_{2n-1}}$ will be denoted by $\omega_1, \cdots, \omega_{2n-1}$, and the fundamental weights in P_{C_n} by $\hat{\omega}_1, \cdots, \hat{\omega}_n$. We can then write

$$P_{A_{2n-1}} = \bigoplus_{i=1}^{2n-1} \mathbb{Z}\omega_i$$

and

$$P_{C_{2n}} = \bigoplus_{i=1}^{n} \mathbb{Z}\hat{\omega}_i,$$

where the direct sums are as $\mathbb{Z}$-modules. Also, for $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ we write $\hat{\lambda} = a_1\hat{\omega}_1 + \cdots + a_n\hat{\omega}_n$ for the corresponding element in P_{C_n}. Weights with non-negative coefficients are called **dominant**. We will denote the corresponding sets by $P_{A_{2n-1}}^+$ and $P_{C_n}^+$. We will consider the real vector spaces $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^\sigma$ spanned by the fundamental weights $\omega_1, \cdots, \omega_{2n-1}$ respectively $\hat{\omega}_1, \cdots, \hat{\omega}_n$. The dominant Weyl chamber is the convex hull of the dominant weights in $\mathfrak{h}_\mathbb{R}^*$ (respectively $(\mathfrak{h}^\sigma)_\mathbb{R}^*$).

3 Tableaux, words and their paths

A **shape** is a finite sequence of non-negative integers $\underline{d} = (d_1, \cdots, d_k)$. An arrangement of boxes of shape $\underline{d}$ consists of $d_1 + \cdots + d_k$ top-left aligned columns such that the first d_1 columns (from right to left) have one box, and the first d_s boxes after the $(d_1 + \cdots + d_{s-1})$-th column have d_s boxes. To a dominant integral weight $\lambda = a_1\omega_1 + \cdots + a_{2n-1}\omega_{2n-1} \in P_{A_{2n-1}}^+$ we assign the shape $\underline{d}_\lambda = (a_1, \cdots, a_{2n-1})$.

Example 1. *For $\lambda = 3\omega_1 + \omega_2$ we have $\underline{d}_\lambda = (3,1)$. To it is associated the following arrangement of boxes:*
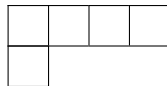
A **semi-standard tableau** of shape $\underline{d}$ is a filling of an arrangement of boxes of shape $\underline{d}$ with letters from the either one of the ordered alphabets

$$\mathscr{A}_{2n} = \{1 < \cdots < 2n\}$$

or

$$\mathscr{C}_n = \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$$

such that entries are strictly increasing downwards along each column of boxes and weakly increasing along each row from left to right. If the entries of a given semi-standard tableau belong to $\mathscr{A}_{2n}$ we will call it a **semi-standard Young tableau**, and if they belong to $\mathscr{C}_n$, we will call it a **symplectic semi-standard tableau**. We will denote the set of semi-standard Young tableau of shape $\underline{d}$ by $\Gamma(\underline{d})^{\text{SSYT}}$.

Example 2. *A semi-standard Young tableau of shape* $(3,1)$*:*

1	*1*	*2*	*5*
2			

Let $\mathscr{W}(\mathscr{A}_{2n})$ denote the word monoid on $\mathscr{A}_{2n}$ and $\mathscr{W}(\mathscr{C}_n)$ be the word monoid on $\mathscr{C}_n$. The **word** $w(\mathscr{T})$ of a semi-standard tableau $\mathscr{T}$ is obtained by reading its rows, from right to left and top to bottom.

Example 3. *The symplectic semi-standard tableau* $\mathscr{T} = $ | *1* | *1* | *2* | *5* | *has word* $w(\mathscr{T}) = 5211\bar{3}$.

4   Restriction of paths that come from words

4.1   Paths and their restrictions

We will consider paths to be continuous maps $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*, \pi' : [0,1] \to (\mathfrak{h}_{\mathbb{R}}^\sigma)^*$ starting at the origin and ending at an integral weight:

$$\pi(0) = 0 = \pi'(0), \ \pi(1) \in P_{A_{2n-1}}, \ \pi'(1) \in P_{C_n}.$$

The map $\mathfrak{h}^* \to (\mathfrak{h}^\sigma)^*, \varphi \mapsto \varphi|_{\mathfrak{h}^\sigma}$ induces a map $\mathrm{res}' : P_{A_{2n-1}} \to P_{C_n}$. Given a path $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$, we define a restricted path $\mathrm{res}(\pi)$ by $\mathrm{res}(\pi) : [0,1] \to (\mathfrak{h}_{\mathbb{R}}^\sigma)^*, t \mapsto \mathrm{res}'(\pi(t))$.

We will also consider the concatenation $\pi_2 * \pi_1$ of two paths π_1 and π_2 with the same codomain: it is the path obtained by translating π_2 to the endpoint $\pi_1(1)$ of π_1. A path is **dominant** if its image is contained in the dominant Weyl chamber.

4.2   Paths that come from words

In the first part of this section we follow Section 2 of [9]. Let $w = w_1 \cdots w_k$ be a word, either in $\mathscr{W}(\mathscr{A}_{2n})$ or in $\mathscr{W}(\mathscr{C}_n)$. To it we assign the path:

$$\pi_w = \pi_{w_k} * \cdots * \pi_{w_1},$$

where, for $w_i \in \mathscr{A}_{2n}$ (respectively $w_i \in \mathscr{C}_n$), the path $\pi_{w_i} : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$ (respectively $\pi_{w_i} : [0,1] \to (\mathfrak{h}_{\mathbb{R}}^\sigma)^*$) is given by $t \mapsto t\varepsilon_{w_i}$, where we define $\varepsilon_{\bar{l}} := -\varepsilon_l$ for $1 \le l \le n$. Also, in general, for paths $\pi_1, \cdots, \pi_k : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$, we have

$$\mathrm{res}(\pi_1 * \cdots * \pi_k) = \mathrm{res}(\pi_1) * \cdots * \mathrm{res}(\pi_k). \tag{4.1}$$

Set $\hat{\varepsilon}_i = \mathrm{res}(\varepsilon_i)$ for $i \in \{1, \cdots, 2n\}$. Then, for $i \in \{1, \cdots, 2n\}$ and $j \in \{1, \cdots, n\}$ we have

$$\hat{\varepsilon}_i(\hat{h}_j) = \begin{cases} 1 & \text{if } i \in \{j, 2n-j\} \\ -1 & \text{if } i \in \{j+1, 2n-j+1\} \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Therefore $\hat{\varepsilon}_i = -\hat{\varepsilon}_{2n-i+1}$, which means we can describe $\mathrm{res}(\pi_w)$ in the following simple way for $w \in \mathscr{W}(\mathscr{A}_{2n})$: First obtain from w a word $\mathrm{res}(w)$ in the alphabet $\mathscr{C}_n$ by replacing a letter w_i in w with $\overline{2n - w_i + 1}$ if $n < w_i \le 2n$. All other letters stay the same. The corresponding path $\pi_{\mathrm{res}(w)}$ coincides with $\mathrm{res}(\pi_w)$ by (4.2) and (4.1).

Example 4. *Let $n = 2$ and $w = 121223341$. Then $\mathrm{res}(w) = 12122\overline{2}\,\overline{2}\,\overline{1}1$.*

5 The Naito-Sagaki conjecture

Let $\lambda \in \mathrm{P}^+_{A_{2n-1}}$ be dominant and let $\mathrm{L}(\lambda)$ be the associated simple module for $\mathfrak{sl}(2n, \mathbb{C})$. Recall the set $\Gamma(\underline{d}_\lambda)^{\mathsf{SSYT}}$ of semi-standard Young tableaux of shape $\underline{d}_\lambda$. Let

$$\mathrm{domres}(\lambda) = \{\eta = \mathrm{res}(\pi_{w(\delta)}) \text{ dominant} : \delta \in \Gamma(\underline{d}_\lambda)^{\mathsf{SSYT}}\}$$

be the set of restricted paths associated to words of elements of $\Gamma(\underline{d}_\lambda)^{\mathsf{SSYT}}$ that are dominant, and for $\nu \in \mathrm{P}^+_{C_{2n}}$, let

$$\mathrm{domres}(\lambda, \nu) = \{\delta \in \mathrm{domres}(\lambda) : \delta(1) = \nu\}.$$

Example 5. *Let $n = 2$ and $\lambda = \omega_1 + \omega_2$. Then*

$$\mathrm{domres}(\lambda) = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}\,,\, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \overline{1} \\ \cline{1-1} \end{array} \right\}$$

$$\mathrm{domres}(\lambda, \lambda) = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

$$\mathrm{domres}(\lambda, \omega_1) = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \overline{1} \\ \cline{1-1} \end{array} \right\}.$$

Theorem 1. *(Schumann-Torres, 2016 [11]) Let $\lambda \in \mathrm{P}^+_{A_{2n-1}}$ be dominant, and let $\mathrm{L}(\lambda)$ be the associated simple module for $\mathfrak{sl}(2n, \mathbb{C})$. Then*

$$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{g}^\sigma}(\mathrm{L}(\lambda)) = \bigoplus_{\delta \in \mathrm{domres}(\lambda)} \mathrm{L}(\delta(\hat{1})).$$

Example 6. *Let $n = 2$ and $\lambda = \omega_1 + \omega_2$ as in Example 5. Then*

$$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{g}^\sigma}(\mathrm{L}(\lambda)) = \mathrm{L}(\hat{\omega}_1) \oplus \mathrm{L}(\hat{\omega}_1 + \hat{\omega}_2).$$

Remark 1. *Theorem 1 was conjectured by Naito-Sagaki in [9] in 2005. In [9] it is stated for $\mathrm{L}(\lambda)$ a representation of $\mathfrak{gl}(2n, \mathbb{C})$ for λ non-negative and dominant. However, the representation of $\mathfrak{gl}(2n, \mathbb{C})$ induced by an irreducible representation of $\mathfrak{sl}(2n, \mathbb{C})$ has the same highest weight and restricts back to itself. See §15.3 in [3].*

6   Littlewood-Richardson tableaux and n-symplectic Sundaram tableaux: branching

Definition 1. *Let $\lambda, \nu \in \mathrm{P}_{A_{2n-1}}$ be two dominant weights such that $\underline{d}_\nu \subset \underline{d}_\lambda$ (this means that one shape is contained in the other when aligned with respect to their top left corners. In Example 7 below we see that $\underline{d}_\nu \subset \underline{d}_\lambda$). A tableau $\mathscr{T}$ of skew shape λ/ν is a filling of an arrangement of boxes of shape λ leaving the boxes that belong to $\nu \subset \lambda$ blank, with the others having entries in the alphabet $\mathscr{A}_{2n}$, and such that these entries are strictly increasing along the columns and weakly increasing along the rows. The word $w(\mathscr{T})$ of $\mathscr{T}$ is obtained just as for semi-standard Young tableaux, reading from right to left and from top to bottom, ignoring the blank boxes.*

A shape $\underline{d} = (d_1, \cdots, d_k)$ is **even** if $d_i = 0$ unless i is an even number, for $i \in \{1, \cdots, k\}$. Also, for a shape $\underline{d}$ define $l(\underline{d})$ to be the length of the longest column of the associated arrangement of boxes.

Definition 2. *Let $\lambda, \nu, \eta \in \mathrm{P}^+_{A_{2n-1}}$ be dominant weights such that the shapes $\underline{d}_\nu$ and $\underline{d}_\eta$ are contained in the shape $\underline{d}_\lambda$ of λ. A **Littlewood-Richardson** (respectively **Littlewood-Richardson Sundaram** or **n-symplectic Sundaram**) tableau of skew shape λ/ν and weight η is a tableau of skew shape λ/ν that is semi-standard, and has a dominant word of weight η (respectively $\underline{d}_\eta$ is even and $2i + 1$ does not appear strictly below row $n + i$ for $i \in \{0, 1, \cdots, \frac{1}{2} l(\underline{d}_\eta)\}$). Here a word $w \in \mathscr{W}(\mathscr{A}_{2n})$ is dominant if the path π_w is dominant. We will denote these skew tableaux by $\mathrm{LR}(\lambda/\nu, \eta)$ (respectively $\mathrm{LRS}(\lambda/\nu, \eta)$). Littlewood-Richardson Sundaram tableaux were introduced by Sundaram in [12], as **n-symplectic** tableaux.*

Remark 2. *Note that if $l(\lambda) \le n$ (such weights are called **stable**) then $\mathrm{LRS}(\lambda/\nu, \eta) = \mathrm{LR}(\lambda/\nu, \eta)$.*

Remark 3. *If λ is stable and $\mathscr{T}$ is a Littlewood-Richardson tableau of skew shape λ/ν then its entries belong to the set $\{1, \cdots, n\}$. This is because if, say, k appears in row l_k of $\mathscr{T}$, then, since the word of $\mathscr{T}$ is dominant, a $k - 1$ must appear either directly above k in the same column, or in a column to the right, and since $\mathscr{T}$ is semi-standard, it appears in at most row $l_k - 1$.*

Example 7. *The tableau $\mathscr{L} =$* *is a Littlewood-Richardson tableau of skew shape λ/ν and weight η for $\lambda = \omega_1 + \omega_2 + \omega_3$, $\nu = \omega_2$, and $\eta = 2\omega_2$ and the tableau $\mathscr{T} =$* *is a Littlewood-Richardson tableau of skew shape λ'/ν' and weight η' for $\lambda' = \omega_3$, $\nu' = \omega_2$, and $\eta' = \omega_1$. Notice that $\mathscr{L}$ is 2-symplectic Sundaram while $\mathscr{T}$ is not.*

Definition 3. *The Littlewood-Richardson coefficient is defined as the number $c^\lambda_{\nu,\eta} \in \mathbb{Z}_{\ge 0}$ such*

that

$$L(\nu) \otimes L(\eta) = \bigoplus_{\underline{d}_\nu \subset \underline{d}_\lambda} c_{\nu,\eta}^\lambda L(\lambda)$$

where $L(\lambda), L(\nu),$ *and* $L(\eta)$ *are all representations of* $\mathfrak{sl}(2n, \mathbb{C})$.

Theorem 2 below is known as the **Littlewood-Richardson rule**. It was first stated in 1943 by Littlewood and Richardson.

Theorem 2. *[2] The Littlewood-Richardson coefficients are obtained by counting Littlewood-Richardson tableaux:*

$$c_{\nu,\eta}^\lambda = |\,\mathrm{LR}(\lambda/\nu, \eta)|.$$

Remark 4. *Theorem 2 implies that* $c_{\nu,\eta}^\lambda = c_{\eta,\nu}^\lambda$.

We will use the notation $c_{\nu,\eta}^\lambda(S) = |\,\mathrm{LRS}(\lambda/\nu, \eta)|$. The following theorem was proven by Sundaram in Chapter IV of her PhD thesis [12]. See also Corollary 3.2 of [13]. For stable weights it was proven by Littlewood in [8] and is known as the Littlewood branching rule.

Theorem 3. *[12] Let* $\lambda \in P_{A_{2n-1}}$ *be dominant. Then*

$$\mathrm{res}_{\mathfrak{g}^\sigma}^{\mathfrak{g}}(L(\lambda)) = \bigoplus_{\substack{\underline{d}_\nu \subset \underline{d}_\lambda \\ l(\underline{d}_\nu) \leq n}} N_{\lambda,\nu} L(\nu)$$

where $N_{\lambda,\nu} = \sum\limits_{\underline{d}_\eta \text{ even}} c_{\nu,\eta}^\lambda(S).$

7 On the proof of the Naito-Sagaki conjecture.

To prove the conjecture, in [11] we construct a bijection

$$\mathrm{domres}(\lambda, \nu) \overset{1:1}{\longleftrightarrow} \bigcup_{\substack{\underline{d}_\eta \subseteq \underline{d}_\lambda; \\ \underline{d}_\eta \text{ even}}} \mathrm{LRS}(\lambda/\eta, \nu). \tag{7.1}$$

To do so we have used combinatorics of "up-down tableaux", and a result by Sundaram and Berele which is in spirit of a symplectic Robinson-Schensted Knuth correspondence. Here we provide a direct bijection in the case of $n = 2$ and $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, for all n. The advantage of this proof is highlighted in Remark 6. The following construction should provide some insight. Given a tableau $\mathscr{T} \in \mathrm{domres}(\lambda)$ we will construct a weight $\eta_\mathscr{T}$ with even shape $\underline{d}_{\eta_\mathscr{T}}$. To do this, first replace, in $\mathscr{T}$, all letters w such

that $n < w \leq 2n$ by $\overline{2n-w+1}$, and denote the resulting symplectic semi-standard tableau by $\text{res}(\mathcal{T})$. Its word $w(\text{res}(\mathcal{T}))$ is equal to the restricted word $\text{res}(w(\mathcal{T}))$. Now, in each column of $\text{res}(\mathcal{T})$, replace an entry w by a blank square if $\overline{w}$ appears to its left in the word $\text{res}(w(\mathcal{T})) = w(\text{res}(\mathcal{T}))$. In that case, replace the entry $\overline{w}$ by a blank square as well. Count the number of blank squares in each column, and order these squares to obtain an arrangement of boxes of shape $\underline{d}_{\eta_{\mathcal{T}}}$.

Example 8. *Let $n = 2$ and*

$$\mathcal{T} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad .$$

Then

$$\text{res}(\mathcal{T}) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \overline{2} \\ \cline{1-1} \end{array} \quad , \ \eta_{\mathcal{T}} = \omega_2, \ \text{and} \ \underline{d}_{\eta_{\mathcal{T}}} = (0,1)$$

where the arrangement of boxes $\begin{array}{|c|}\hline\\\hline\\\hline\end{array}$ *of shape $\underline{d}_{\eta_{\mathcal{T}}}$ is obtained by replacing* $\begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array}$ *in $\text{res}(\mathcal{T})$ by blank squares.*

Lemma 1. *Let $\mathcal{T}$ be as above. Then the shape $\underline{d}_{\eta_{\mathcal{T}}}$ is even.*

Idea of proof. The proof is by induction on the number of right-most aligned columns of $\mathcal{T}$ and is a consequence of the semi-standardness of $\mathcal{T}$. □

Remark 5. *Lemma 1 is only true for $\mathcal{T}$ a semi-standard Young tableau. Consider for example $n = 2$ and the key (as in [4], [14]) $\mathcal{T} = \begin{array}{|c|c|c|}\hline 1 & 4 & 1\\\hline\end{array}$. Then $\text{res}(\mathcal{T}) = \begin{array}{|c|c|c|}\hline 1 & \overline{1} & 1\\\hline\end{array}$ is dominant, however, the shape $\eta_{\mathcal{T}} = (1,0)$ is not even.*

Lemma 2. *If $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ and ν and η are dominant weights in $\text{P}_{A_{2n-1}}$ such that $\underline{d}_\eta$ is even and both $\underline{d}_\eta$ and $\underline{d}_\nu$ are contained in $\underline{d}_\lambda$, then*

$$c^\lambda_{\nu,\eta}(\text{S}) = c^\lambda_{\nu,\eta}.$$

Proof. Assume that $\mathcal{T}$ is a Littlewood-Richardson tableau of skew shape λ/ν and weight η that is not Sundaram. This means that there is at least a "1" in the third row. This is impossible due to semi-standardness of $\mathcal{T}$, the dominance of its word, and $\underline{d}_\eta$ being even. □

Proof of (7.1) *for* $n = 2$ *or* $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$. Let $\mathscr{T} \in \operatorname{domres}(\lambda, \nu)$, and set

$$m_1 = \# \text{ columns in} \mathscr{T} \text{ of the form } \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \bar{1} \\ \hline \end{array},$$

$$m_2 = \# \text{ columns in } \mathscr{T} \text{ of the form } \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \bar{2} \\ \hline \end{array},$$

$$m_3 = \# \text{ columns in } \mathscr{T} \text{ of the form } \begin{array}{|c|} \hline 1 \\ \hline \bar{1} \\ \hline \end{array} \text{ and }$$

$$m_4 = \# \text{ columns in } \mathscr{T} \text{ of the form } \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

If $n = 2$ then $m_2 = m_4$. It follows from semi-standardness and dominance of $\operatorname{res}(w(\mathscr{T}))$ that these are the only possible columns aside from columns of the form $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$ and the single box columns $\boxed{1}$. Note that since $\operatorname{res}(w(\mathscr{T}))$ is dominant, the following condition holds

$$m_1 \leq \lambda_1 - \lambda_2. \tag{7.2}$$

Actually (7.2) is equivalent to the dominance of $\operatorname{res}(w(\mathscr{T}))$, once the λ_i's are set. We assign to $\mathscr{T}$ a Littlewood-Richardson tableau $\varphi(\mathscr{T}) \in \operatorname{LR}(\lambda/\eta_{\mathscr{T}}, \nu)$. By Lemma 1, $\eta_{\mathscr{T}}$ is even.

Write $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3$. Note that $\eta_{\mathscr{T}}$ has $m = m_1 + m_2 + m_3$ columns, all of length 2. Fill in the first $\lambda_1 - m$ right-most boxes in the first row with a "1" , and the first $\lambda_2 - b$ right-most boxes in the second row with a "2". If $n \neq 2$ fill in the first m_4 right-most entries of the third row with a "3". Then fill in the next right-most m_1 entries in the third row with a "2", and the remaining entries with a "1". The resulting tableau $\varphi(\mathscr{T})$ is a Littlewood-Richardson tableau by construction. Now we will show that any element in $\bigcup_{\substack{\underline{d}_\eta \subset \underline{d}_\lambda; \\ \eta \text{ even}}} \operatorname{LR}(\lambda/\eta, \nu)$ can be obtained in this way. Let $\eta \in \operatorname{P}^+_{A_{2n-1}}$ have an even shape $\underline{d}_\eta \subset \underline{d}_\lambda$ (this means $\underline{d}_\eta$ consists of size 2 columns) and let $\mathscr{L} \in \operatorname{LR}(\lambda/\eta, \nu)$. Set

$$l_1 = \# \text{ of 1's in } \mathscr{L}$$
$$l_2 = \# \text{ of 2's in } \mathscr{L}, \text{ and}$$
$$b = \# \text{ of columns of } \eta.$$

Note that this information determines $\mathscr{L}$ together with λ. In view of the previous

construction, we would like to find non-negative integers m_1, m_2, m_3 and m_4 such that

$$l_1 = \lambda_1 - m_1 - m_3 \tag{7.3}$$
$$l_2 = \lambda_2 - m_3 - m_2 \tag{7.4}$$
$$m = m_1 + m_2 + m_3 \tag{7.5}$$
$$m_4 = \text{\# of 3's in } \mathscr{L} \tag{7.6}$$

Since $\mathscr{L}$ has a dominant word, we have

$$l_2 - (\lambda_2 - b) \leq \lambda_1 - \lambda_2. \tag{7.7}$$

Substituting (7.4) and (7.5) in (7.7) we get precisely (7.2), so if we find solutions m_1, m_2, m_3, the resulting tableau will automatically belong to domres(λ, ν). Claim 1 below assures that this is the case.   □

Claim 1. *The system determined by (7.3), (7.4), and (7.5), has integer solutions (possibly zero) b_1, b_2 and b_3 if and only if*

$$m \geq \lambda_1 - l_1 \tag{7.8}$$
$$m \geq \lambda_2 - l_2 \tag{7.9}$$
$$\lambda_1 + \lambda_2 \geq m + l_1 + l_2. \tag{7.10}$$

Idea of proof. The proof is a straightforward solution of the system of equations determined by (7.3) - (7.5).   □

It follows from the definitions that these conditions are satisfied by all elements of

$$\bigcup_{\substack{\underline{d}_\eta \subset \underline{d}_\lambda; \\ \underline{d}_\eta \text{ even}}} \text{LR}(\lambda/\eta, \nu). \tag{7.11}$$

Remark 6. *A first direct consequence of our proof is a description of the left and right hand sides of (7.1) as lattice points of two convex polytopes. In fact, the inequalities and variables used to describe (7.11) figure in Section 3 of [10]. In [11] we exhibit this as a general phenomenon. The approach taken in [11] to prove Theorem 1 is to construct a combinatorial bijection (7.1) as is explained in Section 7. It would be interesting to know if there is a bijection, for all n, (as in Section 7) that comes from a unimodular equivalence.*

Acknowledgements

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